16

ERNST'S EQUATION FOR COLLIDING ELECTROMAGNETIC WAVES

The convenience of exploiting the remarkable analogy between stationary axi-symmetric space-times and colliding plane wave solutions has been thoroughly demonstrated in previous chapters. It has enabled many of the well established solution-generating techniques to be used to obtain new solutions for colliding gravitational waves, and opened up the possibility of finding solutions for arbitrary initial data. The methods used are based on a study of Ernst's equation, which can also be extended (Ernst, 1968*b*) to include electromagnetic fields.

The extension of the Ernst approach to colliding electromagnetic plane waves was first formulated by Chandrasekhar and Xanthopoulos (1985*a*). This approach is described in this chapter, though from a slightly different starting point.

16.1 The field equations

The derivation of Ernst's equations given here is a fairly lengthy one. The purpose is to derive them from the basic field equations given previously in (6.21) and (6.22). We therefore start here with Maxwell's equations in the form (6.21), and make the substitutions

$$\begin{aligned} P_0 &= e^{-U/2}e^{V/2}\sqrt{1+i\sinh W}\;\Phi_0^\circ \\ P_2 &= e^{-U/2}e^{V/2}\sqrt{1-i\sinh W}\;\Phi_2^\circ. \end{aligned} \tag{16.1}$$

With this Maxwell's equations take the form

$$\begin{aligned} P_{2,v} &= \tfrac{1}{2}(1+i\sinh W)(V_v - iW_v\,\mathrm{sech}\,W)P_2 \\ &\quad - \tfrac{1}{2}(1-i\sinh W)(V_u + iW_u\,\mathrm{sech}\,W)P_0 \\ P_{0,u} &= \tfrac{1}{2}(1-i\sinh W)(V_u + iW_u\,\mathrm{sech}\,W)P_0 \\ &\quad - \tfrac{1}{2}(1+i\sinh W)(V_v - iW_v\,\mathrm{sech}\,W)P_2. \end{aligned} \tag{16.2}$$

From this, it is clear that $P_{0,u} = -P_{2,v}$, which implies that there exists a complex potential function $H(u,v)$ such that

$$P_2 = H_u, \qquad P_0 = -H_v. \tag{16.3}$$

Thus

$$\Phi_0^\circ = -\frac{e^{U/2}e^{-V/2}}{\sqrt{1+i\sinh W}}\,H_v, \qquad \Phi_2^\circ = \frac{e^{U/2}e^{-V/2}}{\sqrt{1-i\sinh W}}\,H_u \tag{16.4}$$

and Maxwell's equations reduce to the single equation

$$\begin{aligned} H_{uv} &= \tfrac{1}{2}(1+i\sinh W)(V_v - iW_v\,\mathrm{sech}\,W)H_u \\ &\quad + \tfrac{1}{2}(1-i\sinh W)(V_u + iW_u\,\mathrm{sech}\,W)H_v. \end{aligned} \tag{16.5}$$

It is now convenient to change to the metric functions χ and ω using (11.2) and (11.3), so that the line element takes the form (11.4). In terms of these functions, the main field equations (6.22d,e) now become

$$\begin{aligned} &2\chi_{uv} - U_u\chi_v - U_v\chi_u - \frac{2}{\chi}(\chi_u\chi_v - \omega_u\omega_v) - 2\chi^2e^U(H_u\bar H_v + \bar H_u H_v) = 0 \\ &2\omega_{uv} - U_u\omega_v - U_v\omega_u - \frac{2}{\chi}(\chi_u\omega_v + \chi_v\omega_u) - 2i\chi^2e^U(H_u\bar H_v - \bar H_u H_v) = 0 \end{aligned} \tag{16.6}$$

and Maxwell's equations imply that

$$2\chi H_{uv} + (\chi_v + i\omega_v)H_u + (\chi_u - i\omega_u)H_v = 0 \tag{16.7}$$

where

$$\Phi_0^\circ = -\sqrt{\frac{\chi\,(\chi - i\omega)}{(f+g)\sqrt{\chi^2+\omega^2}}}\,H_v, \qquad \Phi_2^\circ = \sqrt{\frac{\chi\,(\chi + i\omega)}{(f+g)\sqrt{\chi^2+\omega^2}}}\,H_u. \tag{16.8}$$

By way of a digression it may be recalled that, in Chapter 11, it was found convenient to introduce the complex function $Z_\mathrm{o} = \chi + i\omega$. With this, the equations (16.6) may be written as the single complex equation

$$(Z_\mathrm{o} + \bar Z_\mathrm{o})(2Z_{\mathrm{o}uv} - U_uZ_{\mathrm{o}v} - U_vZ_{\mathrm{o}u}) - 4Z_{\mathrm{o}u}Z_{\mathrm{o}v} - (Z_\mathrm{o}+\bar Z_\mathrm{o})^3e^U\bar H_uH_v = 0 \tag{16.9}$$

and (16.7) becomes

$$(Z_\mathrm{o} + \bar Z_\mathrm{o})H_{uv} - Z_{\mathrm{o}v}H_u - \bar Z_{\mathrm{o}u}H_v = 0. \tag{16.10}$$

However, this approach does not seem to lead to anything useful.

It is in fact more convenient at this point to adopt the approach of Chandrasekhar and Ferrari (1984), and to change to the coordinates t

and z using (10.9) to (10.12). In this case, the main field equations (16.6) can be rewritten as

$$\left(\frac{(1-t^2)}{\chi}\chi_t\right)_{,t} - \left(\frac{(1-z^2)}{\chi}\chi_z\right)_{,z} + \frac{(1-t^2)}{\chi^2}\omega_t^2 - \frac{(1-z^2)}{\chi^2}\omega_z^2$$
$$- 2\chi\left(\frac{\sqrt{1-t^2}}{\sqrt{1-z^2}}H_t\bar{H}_t - \frac{\sqrt{1-z^2}}{\sqrt{1-t^2}}H_z\bar{H}_z\right) = 0 \qquad (16.11)$$
$$\left(\frac{(1-t^2)}{\chi^2}\omega_t\right)_{,t} - \left(\frac{(1-z^2)}{\chi^2}\omega_z\right)_{,z} - 2i(H_z\bar{H}_t - H_t\bar{H}_z) = 0.$$

It may be observed that the second of these equations may be written in the form

$$\left(\frac{(1-t^2)}{\chi^2}\omega_t + i(H\bar{H}_z - \bar{H}H_z)\right)_{,t} - \left(\frac{(1-z^2)}{\chi^2}\omega_z + i(H\bar{H}_t - \bar{H}H_t)\right)_{,z} = 0 \qquad (16.12)$$

from which it is clear that, as in (12.34), there exists a real potential function Φ such that

$$\begin{aligned}\Phi_z &= \frac{(1-t^2)}{\chi^2}\,\omega_t + i(H\bar{H}_z - \bar{H}H_z)\\ \Phi_t &= \frac{(1-z^2)}{\chi^2}\,\omega_z + i(H\bar{H}_t - \bar{H}H_t).\end{aligned} \qquad (16.13)$$

At this point, it is convenient to re-introduce the function Ψ of (12.36) defined by

$$\Psi = \sqrt{1-t^2}\sqrt{1-z^2}\,\chi^{-1}. \qquad (16.14)$$

With this, equation (16.13) implies that

$$\begin{aligned}\omega_t &= \frac{(1-z^2)}{\Psi^2}\left(\Phi_z - i(H\bar{H}_z - \bar{H}H_z)\right)\\ \omega_z &= \frac{(1-t^2)}{\Psi^2}\left(\Phi_t - i(H\bar{H}_t - \bar{H}H_t)\right).\end{aligned} \qquad (16.15)$$

These equations are integrable provided

$$\left(\frac{(1-t^2)}{\Psi^2}\left(\Phi_t - i(H\bar{H}_t - \bar{H}H_t)\right)\right)_{,t}$$
$$- \left(\frac{(1-z^2)}{\Psi^2}\left(\Phi_z - i(H\bar{H}_z - \bar{H}H_z)\right)\right)_{,z} = 0. \qquad (16.16)$$

Also, with the substitutions (16.14) and (16.15), (16.11a) becomes

$$\begin{aligned}\left(\frac{(1-t^2)}{\Psi}\Psi_t\right)_{,t} - \left(\frac{(1-z^2)}{\Psi}\Psi_z\right)_{,z}& \\ = -\frac{(1-t^2)}{\Psi^2}\left(\Phi_t - i(H\bar{H}_t - \bar{H}H_t)\right)^2 - \frac{2(1-t^2)}{\Psi^2}H_t\bar{H}_t& \\ + \frac{(1-z^2)}{\Psi^2}\left(\Phi_z - i(H\bar{H}_z - \bar{H}H_z)\right)^2 + \frac{2(1-z^2)}{\Psi^2}H_z\bar{H}_z&\end{aligned} \tag{16.17}$$

and Maxwell's equation (16.7) can be written in the form

$$\begin{aligned}\left(\frac{(1-t^2)}{\Psi}H_t\right)_{,t} - \left(\frac{(1-z^2)}{\Psi}H_z\right)_{,z} = \frac{(1-t^2)}{\Psi^2}H_t(i\Phi_t + H\bar{H}_t - \bar{H}H_t)& \\ - \frac{(1-z^2)}{\Psi^2}H_z(i\Phi_z + H\bar{H}_z - \bar{H}H_z).&\end{aligned} \tag{16.18}$$

It is now convenient to generalize the definition (12.39) and to put

$$Z = \Psi + i\Phi + H\bar{H}. \tag{16.19}$$

With this, equation (16.18) can immediately be written in the form

$$\begin{aligned}(\mathcal{R}e\, Z - H\bar{H})\left\{\left((1-t^2)H_t\right)_{,t} - \left((1-z^2)H_z\right)_{,z}\right\}& \\ = (1-t^2)H_tZ_t - (1-z^2)H_zZ_z& \\ - 2\bar{H}\left\{(1-t^2)H_t^2 - (1-z^2)H_z^2\right\}&\end{aligned} \tag{16.20}$$

and, using (16.20), the main equations in the form (16.16) and (16.17) can be written as the single complex equation

$$\begin{aligned}(\mathcal{R}e\, Z - H\bar{H})\left\{\left((1-t^2)Z_t\right)_{,t} - \left((1-z^2)Z_z\right)_{,z}\right\}& \\ = (1-t^2)Z_t^2 - (1-z^2)Z_z^2& \\ - 2\bar{H}\left\{(1-t^2)H_tZ_t - (1-z^2)H_zZ_z\right\}.&\end{aligned} \tag{16.21}$$

These two equations are Ernst's equations, and are in fact identical to those for stationary axisymmetric Einstein–Maxwell fields. They involve potential functions from which the metric function χ can immediately be obtained using (16.14) and (16.19), and ω can be obtained by integrating (16.15). The electromagnetic field components can then be obtained using (16.8). When $H = 0$, this reduces to (11.18).

It may also be noted that equations (16.20) and (16.21) can be written in the coordinate-independent way

$$\begin{aligned}(\mathcal{R}e\, Z - H\bar{H})\nabla^2 Z &= (\nabla Z)^2 - 2\bar{H}(\nabla H).(\nabla Z)\\ (\mathcal{R}e\, Z - H\bar{H})\nabla^2 H &= (\nabla Z).(\nabla H) - 2\bar{H}(\nabla H)^2.\end{aligned} \tag{16.22}$$

These are the complex Ernst equations for an Einstein–Maxwell field which generalize the form (11.8).

An alternative form of these equations which generalizes (11.19) can be obtained by putting

$$Z = \frac{1+E}{1-E}, \qquad H = \frac{\eta}{1-E}, \tag{16.23}$$

or inversely

$$E = \frac{Z-1}{Z+1}, \qquad \eta = \frac{2H}{Z+1}. \tag{16.24}$$

With these definitions, (16.20) and (16.21) can be rewritten in the form

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})\Big\{\big((1-t^2)E_t\big)_{,t} - \big((1-z^2)E_z\big)_{,z}\Big\}&\\ = -2\bar{E}\left\{(1-t^2)E_t^2 - (1-z^2)E_z^2\right\}&\\ - 2\bar{\eta}\left\{(1-t^2)\eta_t E_t - (1-z^2)\eta_z E_z\right\}&\end{aligned} \tag{16.25}$$

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})\Big\{\big((1-t^2)\eta_t\big)_{,t} - \big((1-z^2)\eta_z\big)_{,z}\Big\}&\\ = -2\bar{\eta}\left\{(1-t^2)\eta_t^2 - (1-z^2)\eta_z^2\right\}&\\ - 2\bar{E}\left\{(1-t^2)\eta_t E_t - (1-z^2)\eta_z E_z\right\}.&\end{aligned} \tag{16.26}$$

These equations can also be written in the coordinate-independent way

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})\nabla^2 E &= -2\nabla E(\bar{E}\nabla E + \bar{\eta}\nabla\eta)\\ (1 - E\bar{E} - \eta\bar{\eta})\nabla^2 \eta &= -2\nabla \eta(\bar{E}\nabla E + \bar{\eta}\nabla\eta).\end{aligned} \tag{16.27}$$

For any solution of (16.20) and (16.21), or equivalently of (16.25) and (16.26), to be acceptable as a colliding plane wave solution, the appropriate boundary conditions must be satisfied. These may be taken in either of the forms (7.15) or (7.16). Alternatively, generalizing the form (11.26), the boundary conditions may be expressed in the form (Griffiths, 1990*a,b*)

$$\begin{aligned}\lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{Z_p\bar{Z}_p - 2(\bar{H}H_p\bar{Z}_p + H\bar{H}_p Z_p) + 2(Z+\bar{Z})H_p\bar{H}_p}{(Z+\bar{Z}-2H\bar{H})^2}\right] &= 2k_1\\ \lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{Z_q\bar{Z}_q - 2(\bar{H}H_q\bar{Z}_q + H\bar{H}_q Z_q) + 2(Z+\bar{Z})H_q\bar{H}_q}{(Z+\bar{Z}-2H\bar{H})^2}\right] &= 2k_2\end{aligned} \tag{16.28}$$

where, for convenience, we have put

$$\frac{\partial}{\partial p}=\frac{\partial}{\partial t}+\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial q}=\frac{\partial}{\partial t}-\frac{\partial}{\partial z} \tag{16.29}$$

and k_1 and k_2 are given by (7.12), and are restricted to the range (7.13).

In terms of the alternative Ernst potential, these conditions can be expressed in the equivalent form

$$\begin{aligned}\lim_{\substack{t\to0\\z\to0}}\left[\frac{(1-\eta\bar\eta)E_p\bar E_p+\bar\eta E\eta_p\bar E_p+\eta\bar E\bar\eta_pE_p+(1-E\bar E)\eta_p\bar\eta_p}{(1-E\bar E-\eta\bar\eta)^2}\right]&=2k_1\\ \lim_{\substack{t\to0\\z\to0}}\left[\frac{(1-\eta\bar\eta)E_q\bar E_q-\bar\eta E\eta_q\bar E_q-\eta\bar E\bar\eta_qE_q+(1-E\bar E)\eta_q\bar\eta_q}{(1-E\bar E-\eta\bar\eta)^2}\right]&=2k_2.\end{aligned} \tag{16.30}$$

Having found an acceptable solution for Z and H, or alternatively for E and η, it is finally necessary to determine the remaining metric function M. This can be achieved by integrating equations (7.9) in an appropriate form and using (7.8).

16.2 A simple class of solutions

It has been observed by Chandrasekhar and Xanthopoulos (1987*a*), that equation (16.21) is automatically satisfied when Z is a constant. It may also be noticed that, in this case, the remaining equation (16.20) does not contain the imaginary part of Z, and a constant imaginary part of Z would only yield a constant Φ which would have no effect on the metric through (16.15). It is therefore possible to consider Z as a real constant. In addition, replacing Z by $\alpha^2 Z$ and H by αH, where α is a real constant, leaves (16.20) unchanged. Thus, once it is assumed that Z is a constant, no loss of generality is entailed by the assumption that

$$Z=1. \tag{16.31}$$

In this case (16.20) becomes

$$\begin{aligned}(1-H\bar H)\Big(\big((1-t^2)H_t\big)_{,t}-\big((1-z^2)H_z\big)_{,z}\Big)\\ =-2\bar H\left((1-t^2)H_t^2-(1-z^2)H_z^2\right)\end{aligned} \tag{16.32}$$

which is identical in form to (11.19).

Thus the assumption (16.31) requires that H satisfies the alternative form of the Ernst equation for a vacuum. This enables numerous solutions

to be easily generated. All that is required is a solution E_o of the vacuum Ernst equation (11.19). Then, a new electromagnetic solution is obtained simply by putting

$$Z = 1, \qquad H = E_o. \tag{16.33}$$

Expressing this in terms of the alternative Ernst functions, if E_o is a solution of the vacuum Ernst equation (11.14), then a new electromagnetic solution of (16.25,26) is obtained simply by putting

$$E = 0, \qquad \eta = E_o. \tag{16.34}$$

Moreover, by comparing (16.28) with (11.27) it is clear that, if the boundary conditions for colliding waves are satisfied for the vacuum solution E_o, then the appropriate boundary conditions are automatically satisfied for the new solution (16.33).

With (16.33), it follows that

$$\Psi = 1 - E_o\bar{E}_o, \qquad \Phi = 0, \qquad \chi = \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{1 - E_o\bar{E}_o} \tag{16.35}$$

and ω can be found by integrating (16.15), which may be rewritten as

$$\begin{aligned} \omega_t &= -i\frac{(1-z^2)}{(1-E_o\bar{E}_o)^2}(E_o\bar{E}_{oz} - \bar{E}_o E_{oz}) \\ \omega_z &= -i\frac{(1-t^2)}{(1-E_o\bar{E}_o)^2}(E_o\bar{E}_{ot} - \bar{E}_o E_{ot}). \end{aligned} \tag{16.36}$$

The remaining metric function M may finally be determined by integrating an appropriate transformation of equations (7.9). However, the resulting equations will not be a generalization of (11.21), since in that equation the function E was related to χ and ω rather than Ψ and Φ. In this case (7.9) may be written in the form

$$\begin{aligned} S_f &= -\frac{1}{2(f+g)} - \frac{E_o\bar{E}_{of} + \bar{E}_o E_{of}}{(1-E_o\bar{E}_o)} - \frac{2(f+g)E_{of}\bar{E}_{of}}{(1-E_o\bar{E}_o)^2} \\ S_g &= -\frac{1}{2(f+g)} - \frac{E_o\bar{E}_{og} + \bar{E}_o E_{og}}{(1-E_o\bar{E}_o)} - \frac{2(f+g)E_{og}\bar{E}_{og}}{(1-E_o\bar{E}_o)^2}. \end{aligned} \tag{16.37}$$

It may be recalled that E_o is any solution of the vacuum Ernst equation (11.14). Thus, for any initial vacuum solution expressed in terms of the potentials Ψ_o and Φ_o, an electromagnetic solution can be obtained by the above method using

$$E_o = \frac{\Psi_o + i\Phi_o - 1}{\Psi_o + i\Phi_o + 1}, \qquad \Psi = \frac{4\Psi_o}{(\Psi_o+1)^2 + \Phi_o^2}, \qquad \Phi = 0. \tag{16.38}$$

In addition, (16.37) may now be written in the form

$$\begin{aligned} S_f &= -\frac{1}{2(f+g)} - \frac{\Psi_f}{\Psi} - \frac{(f+g)({\Psi_{\rm o}}_f^{\,2} + {\Phi_{\rm o}}_f^{\,2})}{2\Psi_{\rm o}^2} \\ S_g &= -\frac{1}{2(f+g)} - \frac{\Psi_g}{\Psi} - \frac{(f+g)({\Psi_{\rm o}}_g^{\,2} + {\Phi_{\rm o}}_g^{\,2})}{2\Psi_{\rm o}^2}. \end{aligned} \tag{16.39}$$

By comparing this with (12.41) it may be seen that, if the initial vacuum solution contains the metric function $M_{\rm o}$, then equations (16.39) can be integrated to give

$$e^{-M} = \frac{\Psi_{\rm o}}{\Psi} e^{-M_{\rm o}} = \tfrac{1}{4}\big((\Psi_{\rm o}+1)^2 + \Phi_{\rm o}^2\big)e^{-M_{\rm o}}. \tag{16.40}$$

It also follows from this that

$$U = U_{\rm o} \tag{16.41}$$

only if $\big((\Psi_{\rm o}+1)^2 + \Phi_{\rm o}^2\big)$ is continuous across the boundaries of region IV.

16.3 The Bell–Szekeres solution

In order to describe the Bell–Szekeres solution in the present notation, it may immediately be observed by substituting (15.6) into (16.4), and using (11.2) that

$$H = \sin(au - bv), \qquad \chi = \frac{\cos(au+bv)}{\cos(au-bv)}, \qquad \omega = 0. \tag{16.42}$$

From (16.14), it follows that $\Psi = \cos^2(au - bv)$, and clearly $\Phi = 0$. The Ernst potentials for the Bell–Szekeres solution are therefore given by

$$Z = 1, \qquad H = z. \tag{16.43}$$

The alternative Ernst potentials are

$$E = 0, \qquad \eta = z. \tag{16.44}$$

This clearly satisfies the boundary conditions (16.28) with $k_1 = k_2 = \frac{1}{2}$ as required.

It has frequently been noted that V can always be replaced by $-V$. Such a transformation, in this case, would yield a solution in which

$$H = \sin(au + bv), \qquad \chi = \frac{\cos(au-bv)}{\cos(au+bv)}, \qquad \omega = 0 \tag{16.45}$$

and therefore

$$H = t, \qquad Z = 1, \qquad \Phi_0^\circ = -b, \qquad \Phi_2^\circ = a. \tag{16.46}$$

The potentials (16.43) and (16.46) seem to indicate that the Bell–Szekeres solution may be included in a solution obtained by the method of the previous section, in which $Z = 1$ and H is taken to be the Ernst function (13.4) of the Nutku–Halil solution: namely

$$H = pt + iqz \tag{16.47}$$

where $p^2 + q^2 = 1$. It may be noted that, when $p = 0$, a change from $H = z$ to $H = iz$ in (16.43) would yield $\Phi_0^\circ = ib$ and $\Phi_2^\circ = ia$ which are related to the components of (15.6) by a simple duality rotation.

The potential (16.47), with $Z = 1$, implies that

$$\Psi = 1 - p^2t^2 - q^2z^2, \qquad \Phi = 0, \qquad \chi = \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{1 - p^2t^2 - q^2z^2} \tag{16.48}$$

and equations (16.36) can be integrated to give

$$\omega = \frac{p}{q}\frac{(1-t^2)}{(1 - p^2t^2 - q^2z^2)} + \text{const.} \tag{16.49}$$

This solution has been considered in detail by Chandrasekhar and Xanthopoulos (1987*a*), who have taken the constant of integration in (16.49) to be zero so that $\omega = 0$ when $t = 1$. They have then shown that the resulting solution is a simple transformation of the conformally flat Bell–Szekeres solution.

Alternatively, taking the constant of integration in (16.49) to be $-p/q$ gives the rotation of the Bell–Szekeres solution described in Section 15.4, with the parameters related to that of (15.18) by

$$p = \cos\frac{\alpha}{2}, \qquad q = -\sin\frac{\alpha}{2}. \tag{16.50}$$

In either case, it may be concluded that the solution arising from (16.47) is the Bell–Szekeres solution, but with the polarization of the step electromagnetic waves being non-aligned.

It may also be observed that the general solution above can be obtained by the methods of the previous section with the initial vacuum solution given by

$$E_\text{o} = \frac{\Psi_\text{o} + i\Phi_\text{o} - 1}{\Psi_\text{o} + i\Phi_\text{o} + 1} = pt + iqz \tag{16.51}$$

which is equivalent to the Ernst potential (13.10) that gives rise to the vacuum Chandrasekhar–Xanthopoulos solution. On substituting these expressions, (16.40) takes the form

$$e^{-M} = \frac{1}{(1-pt)^2 + q^2z^2} e^{-M_\circ} \tag{16.52}$$

which, from (13.20), becomes

$$e^{-M} = \frac{1}{\sqrt{1-u^2}\sqrt{1-v^2}}. \tag{16.53}$$

It follows from this that $U = U_\circ = -\log(1-u^2-v^2)$. However, it is clearly appropriate to make the transformation

$$u \to \sin au, \qquad v \to \sin bv \tag{16.54}$$

which, with a scale factor, gives the familiar form of the Bell–Szekeres solution.